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THE CUTOFF $\lambda\phi^4$ O(N) MODEL IN THE LARGE N LIMIT

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Abstract

A cutoff version of the $\lambda\phi^4$ O(N) model is considered to leading order in 1/N with particular attention to the effective potential, which is surprisingly rich in structure. With suitable restriction on a background classical field, one finds a phenomenologically viable model with spontaneously broken symmetry, a potential bounded below, and amplitudes free of tachyons. The model has an O(N–1) singlet resonance in both weak and strong coupling, which can be interpreted as the Higgs meson in applications. Further, an unphysical resonance, which can be used to define a triviality scale for the model, appears at a mass above the cutoff mass Λ. The phomelogical aspects of our discussion are consistent with previous studies of closely related models by Heller, *et al.*

The question of the double-scaling limit for the cutoff model is considered as an application of the effective potential. It is shown that the double-scaling limit is not possible.

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I. Introduction

The possibility of a strongly interacting Higgs sector, with large Higgs mass, suggests the need for methods which can deal with the interactions of scalars that are non-perturbative in the coupling constant. One approach to this issue is to reorganize the theory in terms of some other expansion parameter, such as the 1/N expansion for a theory with internal symmetry such as O(4), continued to O(N) for example. Predictions are then obtained by evaluating results of the expansion at the physical value of N (N=4 say). The 1/N expansion for $\lambda\phi^4$ theory (in 3+1 dimensions) with O(N) symmetry (the so-called vector model) has been extensively studied as a renormalized field theory [1, 2, 3, 4]. However, the renormalized vector model encounters a number of problems [1, 2]. Among these are:

1) The effective potential of the theory is double valued [2], with the lower energy branch of the potential describing a phase of the theory with <u>unbroken</u> internal symmetry, *i.e.,* $< \Phi_a >= 0$. This phase is tachyon free in all orders of the 1/N expansion. The higher energy branch of the effective potential does allow a spontaneously broken symmetry. However, this phase contains tachyons, presumably as a symptom of decay to the lower energy phase. In higher orders of the 1/N expansion, the higher energy branch of the effective potential becomes everywhere complex.

2) The effective potential has no lowest energy bound as the external field $\phi \rightarrow \infty$ [1, 2, 3]. The tachyon-free phase (*i.e.,* with $< \Phi_a >= 0$) of the 1/N expansion tunnels non-perturbatively to this unstable vacuum, with an amplitude proportional to exp(–N).
3) Most importantly, it is widely believed [3, 5] that $\lambda\phi^4$ theory in 4–dimensions is actually a trivial, free-field theory, which may be at the root of the problems summarized above.
These difficulties would seem to make the renormalized vector model, evaluated in the 1/N expansion, unsuitable for phenomenology.

One possible way to deal with these problems is to consider a cutoff version of the vector model in the 1/N expansion [6]. One introduces a cutoff Λ into the theory, which represents a mass-scale above which the self-coupled Higgs sector can no longer be considered isolated from other essential degrees of freedom of a more complete theory. There are other possible interpretations of the cutoff, particularly if the Higgs is not elementary, but only represents a scalar bound-state of some effective field theory. However, one does not need to commit oneself to any particular physical interpretation of the cutoff in this paper.

A number of results for the cutoff vector model in the large N limit are available [6, 7, 8], but no systematic study of the effective potential of the model has been undertaken. In this paper we give a careful presentation of the effective potential to leading order in 1/N for the cutoff vector model. This is the main new contribution of this paper. We find that the effective potential is surprisingly rich in structure, with several phases possible, depending on the parameters of the model. Some restrictions on the parameters and external field strengths will be required to obtain a phenomenologically viable model for energies $< \Lambda$. The nature of these restrictions will become clear after our analysis of the effective potential. With such restrictions we find a phenomenologically viable model with spontaneously broken symmetry, a potential bounded below, and amplitudes

free of tachyons. We then study the resonance structure of meson-meson scattering in the O(N–1) singlet sector in this phase. The scattering amplitude exhibits a resonance which can be interpreted as the Higgs meson of the model, both in weak and in strong coupling, as well as an unphysical resonance above the cutoff mass Λ, which we interpret as defining the triviality mass-scale of the theory.

In Sec. 2 we formulate the cutoff version of the vector model, while Sec. 3 presents the effective potential of the model. An analysis of the Green's functions of the model in Sec. 4 provides criteria for the presence or absence of tachyons. Sec. 5 considers the O(N) scalar bound-state structure in the phase with spontaneously broken symmetry. This allows us to study the dependence of a Higgs meson mass on the coupling constant and cutoff of the model. In Sec. 6 we consider the possibility of a double-scaling limit for this cutoff model. Some important observations about cutoff models are included in the concluding Sec. 7.

II. The Cutoff Model

We formulate the cutoff vector model using the conventions and notation of Abbott, *et al.,* ref. [2]. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}\,\mu_0^2 \Phi^2 - \frac{\lambda_0}{4!N}\,(\Phi^2)^2 \tag{2.1}$$

where Φ_a ($a = 1$ to N) is an N-component quantum field, and $\Phi^2 = \sum_{a=1}^{N} \Phi_a \Phi_a$. In (2.1) μ_0 and λ_0 are the bare mass and coupling constant, respectively. To leading order in 1/N the effective potential satisfies

$$\frac{\partial V(\phi^2)}{\partial \phi_a} = \chi\,\phi_a \tag{2.2}$$

with χ related to the (constant) classical field ϕ_a by the gap equation

$$\chi = \mu_0^2 + \frac{1}{6}\,\lambda_0 \left(\frac{\phi^2}{N}\right) + \frac{1}{6}\,\lambda_0 \int \frac{d^4 k}{(2\pi)^4}\,\frac{1}{k^2 + \chi} \tag{2.3}$$

where the integral is over Euclidean momenta. In order to make the gap equation well-defined, the integration in (2.3) is cutoff at Λ. With fixed cutoff, both μ_0 and λ_0 are finite. Throughout this paper we regard μ_0, λ_0 and Λ as fixed, finite parameters of the theory. In particular, we will <u>not</u> renormalize the theory, and will <u>not</u> "run" the cutoff Λ in the sense of the renormalization group. For us Λ is a fixed energy above which the <u>pure</u> $\lambda\phi^4$ no longer should be considered as an isolated theory. As a consequence the gap equation becomes

$$\chi = \mu_0^2 + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N}\right) + \left(\frac{\lambda_0}{96\pi^2}\right)\left[\Lambda^2 + \chi \log\left(\frac{\chi}{\Lambda^2 + \chi}\right)\right] \tag{2.4}$$

It is important to note that we do <u>not</u> assume $(\chi/\Lambda^2) \ll 1$, contrary to what is frequently done. One can use (2.2) and (2.4) to find the effective potential

$$V \;=\; -\frac{3}{2}\,\frac{N}{\lambda_0}\,\chi^2 \;+\; \frac{1}{2}\,\chi\phi^2 \;+\; \frac{3N\mu_0^2}{\lambda_0}\,\chi$$

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$$-\frac{N}{64\pi^2}\left[\Lambda^4\log\left(\frac{\Lambda^2}{\Lambda^2+\chi}\right)-\chi^2\log\left(\frac{\chi}{\Lambda^2+\chi}\right)-\chi\Lambda^2\right] \tag{2.5}$$

It is easy to extract some general properties of the cutoff theory from (2.2), (2.4) and (2.5). Observe that for $\lambda_0 > 0$

$$\chi \xrightarrow[\operatorname{Re}\chi\to+\infty]{} \frac{\lambda_0\phi^2}{6N} + \mathcal{O}(1) \;, \tag{2.6}$$

$$\operatorname{Re}\frac{\partial V}{\partial\phi^2} \xrightarrow[\operatorname{Re}\chi\to+\infty]{} \frac{\lambda_0\phi^2}{12N} + \mathcal{O}(1) \;, \tag{2.7}$$

so that

$$\operatorname{Re}V(\phi^2) \xrightarrow[\operatorname{Re}\chi\to+\infty]{} \frac{\lambda_0\phi^4}{24N} + \mathcal{O}(\phi^2) \;. \tag{2.8}$$

Therefore the effective potential has a lower-bound for $\lambda_0 > 0$ when $\operatorname{Re}\chi \to +\infty$. The behavior exhibited by the cutoff theory in (2.6)–(2.8) is in marked contrast to that of the renormalized vector model, where [2]

$$\operatorname{Re}\chi \underset{\phi^2\to\infty}{\widetilde{}} \frac{-16\pi^2(\phi^2/N)}{\ln(\phi^2/M^2)}$$

$$\operatorname{Re}\frac{\partial V(\phi^2)}{\partial\phi^2} \underset{\phi^2\to\infty}{\widetilde{}} \frac{-8\pi^2(\phi^2/N)}{\ln(\phi^2/M^2)}$$

(renormalized theory)

independent of the parameters of the theory, but with M^2 a renormalization mass. [These are eqs. (2.14) and (2.15) of Abbott, *et al.* [2]] A comparison of (2.6)–(2.8) with the analogous equations of the renormalized theory (shown above) emphasizes the dramatic difference between the behavior of the effective potential in the two cases. In the cutoff theory, the effective potential with $\lambda_0 > 0$ behaves classically as $\operatorname{Re}\chi \to +\infty$, while in the renormalized theory, the effective potential is dominated by quantum effects in this limit.

From (2.5) we observe that the effective potential is everywhere complex for $\chi < -\Lambda^2$, and that

$$\operatorname{Re}V \underset{\chi\to-\Lambda^2_-}{\widetilde{}} -\infty \;. \tag{2.9}$$

As a result of this behavior, we will eventually restrict the classical external fields so that $\chi > -\Lambda^2$. Before imposing such a constraint, we first study (in the next section) the effective potential without restrictions on χ or the sign of λ_0. The behavior of the effective potential in this unrestricted case will be instructive, as it will underline the limitations of the model. We will argue that it is plausible to require that $\lambda_0 > 0$ and $\chi > -\Lambda^2$, and then will study the consequences of the resulting restricted model.

Since the most important application of the model is to problems with spontaneous symmetry breaking, we end this section by focusing on the conditions for this to occur. At

a stationary point of the effective potential, the right-hand side of (2.2) must vanish. If the classical field at this point satisfies $\phi_a =< \Phi_a >\neq 0$, then [1, 2]

$$\chi = 0 \tag{2.10}$$

for spontaneous symmetry breaking. Combining (2.10) with (2.4) implies that

$$\phi^2 =< \Phi_a >^2 = -N \left(\frac{6\mu_0^2}{\lambda_0} + \frac{\Lambda^2}{16\pi^2} \right) > 0 \tag{2.11}$$

at the minimum of the potential. Since in our three parameter model, μ_0, λ_0 and Λ are finite, spontaneous symmetry breaking requires

$$\left(\frac{\mu_0^2}{\lambda_0} \right) < - \left(\frac{\Lambda^2}{96\pi^2} \right) < 0 \ . \tag{2.12}$$

For convenience define the auxiliary quantity

$$\frac{\mu^2}{\lambda} = \frac{\mu_0^2}{\lambda_0} + \frac{\Lambda^2}{96\pi^2} \ . \tag{2.13}$$

We call (μ^2/λ) a "dressed" parameter, in contrast with the usual renormalized parameters of renormalized theories. That is, since Λ is finite, the bare quantities are dressed by interactions, even though the bare quantities are themselves finite. It is convenient to define

$$v_0^2 = \frac{-6N\mu_0^2}{\lambda_0} \tag{2.14a}$$

and

$$v^2 = \frac{-6N\mu^2}{\lambda} \tag{2.14b}$$

so that

$$< \Phi_a >^2 = v^2 = v_0^2 - \frac{N\Lambda^2}{16\pi^2} > 0 \tag{2.15}$$

when the O(N) symmetry is spontaneously broken. [From (2.12) we find the usual condition, $\mu_0^2/\lambda_0 < 0$ for spontaneous symmetry breaking.]

III. The Effective Potential

In this section we discuss the effective potential of the cutoff vector model first without placing any restrictions on χ or λ_0. Our analysis will show that the unrestricted cutoff model is inconsistent, as the effective potential has no lower bound. As a result, we will argue that $\lambda_0 > 0$ and $\chi > -\Lambda^2$ is required if the model is to be viable as an approximate description of a strongly interacting scalar sector.

The effective potential $V(\phi)$ can be evaluated from (2.5), using the relation between χ and ϕ^2 given by (2.4). The analysis is straightforward, and follows the strategy of Abbott, *et al.* [2]. In so doing, one must carefully examine (2.4) and (2.5) for multiple branches of the

potential. Since the analysis is a bit tedious, and the details not particularly illuminating, we present only the results in a series of figures.

A.   Unrestricted model

Results for the case $\lambda_0 > 0$, together with $(\frac{6\mu^2 N}{\lambda}) < 0$ [for small and large values of the coupling λ_0 respectively] are presented in Figs. 1 and 2. Fig. 3 gives $V(\phi^2)$ for $\lambda_0 > 0$ and $(\frac{6\mu^2 N}{\lambda}) > 0$. In Figs. 4 and 5, $V(\phi^2)$ is displayed for $\lambda_0 < 0$ with $(\frac{6\mu^2 N}{\lambda}) < 0$ and $(\frac{6\mu^2 N}{\lambda}) > 0$ respectively. Dotted lines in the figures indicate Im $V \neq 0$.

We observe in Figs. 1 to 5 that in each case the effective potential is multivalued, and has no lowest energy bound. In each of the figures, branch II, which corresponds to $\chi < -\Lambda^2$, is everywhere complex, and lies below branch I for large ϕ^2. Note that branch I of our Figs. 4 and 5 are qualitatively similar to Figs. 2 and 3 of Abbott *et al.* On the other hand branch I of our Figs. 1 to 3 behave qualitatively like that of the classical theory. In summary, we conclude from Figs. 1 to 5, that the cutoff vector model is not consistent if no restrictions are placed on the model. [See Sec. 7 for further comments on the consistency of cutoff-models.]

B.   Restricted model

We impose the restriction

$$\chi > -\Lambda^2 \tag{3.1}$$

on the external fields. Clearly (3.1) eliminates branch II of $V(\phi^2)$ from consideration. Branch II represents an instability which occurs when the classical field χ becomes too strong. [It is useful to remember that Λ^2 is a Euclidean cutoff.] We impose (3.1) and show $V(\phi^2)$ in Figs. 6 and 7 for $\lambda_0 > 0$ and $6\mu^2 N/\lambda < 0$ and $6\mu^2 N/\lambda > 0$ respectively. Figs. 8 and 9 give analogous results for $\lambda_0 < 0$. Notice that now $V(\phi)$ is single valued for $\lambda_0 > 0$, in contrast to the renormalized theory and is similar to that of the classical theory, while for $\lambda_0 < 0$ the effective potential qualitatively resembles the effective potential of Abbott, *et al.* [However, in the cutoff model Re $V(\phi^2) \xrightarrow[\phi^2 \to \infty]{} \frac{-|\lambda_0|\phi^4}{24N}$ when $\lambda_0 < 0$, while in the renormalized theory Re$V(\phi^2) \sim \frac{-4\pi^2\phi^4}{\ln \phi^2}$ for all values of the coupling constant.]

In the next section we consider whether tachyons are present in the restricted model.

IV.   Green's Functions and Tachyons

In this section we consider the $\phi - \chi$ inverse propagator for the cutoff vector model, and impose the restriction $\chi > -\Lambda^2$ on the external classical fields. Figures 6 to 9 describe the effective potential for this case. As noted in Sec. 3, when $\lambda_0 < 0$ the effective potential has features which are qualitatively similar to that of Abbott, *et al.*, [*c.f.* their figs. 2 and 3]. Therefore, for reasons discussed in the Introduction and ref. [2], we consider the cutoff model with $\lambda_0 < 0$ unsuitable for phenomenology. Therefore we focus on the cutoff model with $\chi > -\Lambda^2$ and $\lambda_0 > 0$ for the bulk of the paper. We return to the question of $\lambda_0 < 0$ when we consider the possibility of a double-scaling limit in Sec. 6.

The $\Phi - \chi$ matrix inverse propagator in the presence of external fields ϕ and χ is [c.f. ref. [2], eqn. (4.2)].

$$D^{-1}(-k^2, \phi, \chi) = \begin{bmatrix} (k^2 + \chi)\delta_{ab} & \phi_a \\ \phi_b & -3N\left(\frac{1}{\lambda_0} - \bar{B}(\chi, k^2, \Lambda^2)\right) \end{bmatrix} \tag{4.1}$$

for Euclidean momenta k^2. The function

$$\bar{B}(\chi, k^2, \Lambda^2) = -\frac{1}{6} \int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + \chi)[(k+p)^2 + \chi]}$$
$$< 0 \tag{4.2}$$

is given by an integral over Euclidean momenta, and is real for $\chi > -\Lambda^2$. The fact that the "bubble" integral (4.2) is real for $\chi > -\Lambda^2$ is an important aspect of the consistency of our approach, since this is the same requirement for an acceptable effective potential. The integration is made finite by a cutoff Λ, with some ambiguity in specifying a cutoff. For simplicity we use a sharp-cutoff in p-space. We have verified by explicit calculation, that several different cutoff methods give substantially the same result. In practice this is not a problem. The essential point is that all reasonable cutoffs insure that $\bar{B} < 0$ for Euclidean k^2. This point was also emphasized in ref. 17. [By contrast, in the renormalized theory, $\bar{B}(k^2)$ need not have a definite sign, as it is made finite by a subtraction.] On carrying out the integration in (4.2) with a sharp cutoff, we find

$$\bar{B}(\chi, k^2, \Lambda) = -\frac{1}{96\pi^2}\left\{ \ln\left(\frac{\Lambda^2 + \chi}{\chi}\right) + \right.$$

$$+ 2\sqrt{\frac{k^2 + 4(\Lambda^2 + \chi)}{k^2}} \left[1 - \frac{2\Lambda^2}{k^2 + 4(\Lambda^2 + \chi)}\right] \ln\left[\frac{\sqrt{k^2} + \sqrt{k^2 + 4(\Lambda^2 + \chi)}}{2\sqrt{\Lambda^2 + \chi}}\right]$$

$$\left. - 2\sqrt{\frac{k^2 + 4\chi}{k^2}} \ln\left[\frac{\sqrt{k^2} + \sqrt{k^2 + 4\chi}}{2\sqrt{\chi}}\right]\right\} . \tag{4.3}$$

This expression differs from Abbott, *et al.*, since we have assumed neither $\chi \ll \Lambda^2$ nor $k^2 \ll \Lambda^2$. This will be important in what follows.

A. Unbroken Symmetry

The O(N) symmetry is unbroken if $< \Phi_a > = \phi_a = 0$. Since $\chi > 0$, $\chi = m^2$ becomes the physical meson mass, and

$$D^{-1}(-k^2, 0, m^2) = \begin{bmatrix} (k^2 + m^2)\delta_{ab} & 0 \\ 0 & -3N\left(\frac{1}{\lambda_0} - \bar{B}(m^2, k^2, \Lambda^2)\right) \end{bmatrix} \tag{4.4}$$

is the $\Phi - \chi$ inverse propagator for this case.

When $\chi > -\Lambda^2$, $\bar{B}(m^2, k^2, \Lambda^2) < 0$ and real for k^2 Euclidean. Therefore from (4.2)

$$\left[\frac{1}{\lambda_0} - \bar{B}(m^2, k^2, \Lambda^2) \right] > 0 \tag{4.5}$$

for $\lambda_0 > 0$, independent of the details of the cutoff. Hence, D^{-1} does not vanish for Euclidean k^2, and tachyons are absent from the model in this phase which is as expected from the effective potential, Fig. 7.

B.   Spontaneously Broken Symmetry

The O(N) symmetry breaks spontaneously to O(N–1) if

$$\langle \Phi_a \rangle^2 \;\; = \;\; \phi^2 \;\; = \;\; v^2$$

$$= \;\; - \frac{6N\mu^2}{\lambda} \;\; > \;\; 0 \tag{4.6}$$

and $\chi = 0$. The existence of tachyons depends on whether $\det D^{-1}(-k^2, v, 0)$ vanishes for Euclidean k^2, $i.e.$, if

$$\det D^{-1}(-k^2, v, 0)$$

$$= k^2 \left\{ -3N \left[\frac{1}{\lambda_0} - \bar{B}(0, k^2, \Lambda^2) \right] \right\} - v^2 \stackrel{?}{=} 0 \tag{4.7}$$

However, $-\bar{B}(0, k^2, \Lambda^2) > 0$ for k^2 Euclidean, independent of the details of the cutoff. Therefore for $\lambda_0 > 0$ and $v^2 > 0$, tachyons are absent from this phase, in accord with the effective potential described by Fig. 6. In fact

$$\bar{B}(0, k^2, \Lambda^2) = -\frac{1}{96\pi^2} \left\{ \ln\left(\frac{\Lambda^2}{k^2}\right) + 2\sqrt{\frac{k^2 + 4\Lambda^2}{k^2}} \left[1 - \frac{2\Lambda^2}{(k^2 + 4\Lambda^2)} \right] \ln\left[\frac{\sqrt{k^2} + \sqrt{k^2 + 4\Lambda^2}}{2\Lambda} \right] \right\} \tag{4.8a}$$

$$\xrightarrow[k^2 \gg \Lambda^2]{} \quad - \frac{1}{96\pi^2} \left\{ 2\left(\frac{\Lambda^2}{k^2}\right) + \mathcal{O}(\Lambda^4/k^4) \right\} < 0 \tag{4.8b}$$

$$\xrightarrow[k^2 \ll \Lambda^2]{} \quad - \frac{1}{96\pi^2} \left\{ \ln\left(\frac{\Lambda^2}{k^2}\right) + 1 + \mathcal{O}(k^2/\Lambda^2) \right\} < 0 \tag{4.8c}$$

[Notice that (4.8c) changes sign, if it is naively used for $k^2 \gg \Lambda^2$. This unjustifiable use of (4.8c) for large k^2 violates the general requirement that $\bar{B} < 0$. If (4.8c) were to be used for all k^2, one would conclude erroneously that there was a tachyon in this phase.]

We conclude that the cutoff vector model with $\lambda_0 > 0$ and $\chi > -\Lambda^2$ is free of tachyons to leading order in 1/N. It would be interesting to see if this feature persists in higher orders of 1/N.

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C. $\lambda_0 < 0$

We briefly summarize the issue of tachyons for $\lambda_0 < 0$. The effective potential is given by Figs. 8 and 9 for our model restricted to $\chi > -\lambda^2$. Notice the similarity of our Figs. 8 and 9 to the effective potentials found by Abbott, *et al.* [2] in their Figs. 2 and 3. When $\lambda_0 < 0$ and $(\mu^2/\lambda) < 0$, (4.7) is the appropriate equation for this issue. If $\langle \Phi \rangle^2 > 0$, what is relevant is the upper-branch of Fig. 8. It is straightforward to show that tachyons are <u>always</u> present for vacuua chosen on the upper branch of Fig. 8. Similarly one can ask whether tachyons are present when $\lambda_0 < 0$ and $(\mu^2/\lambda) > 0$. [Figure 9 is now appropriate.] Tachyons will be present if (for $\lambda_0 < 0$)

$$- \bar{B}(m^2, \ k^2, \ \Lambda^2) \stackrel{?}{=} \left| \frac{1}{\lambda_0} \right| . \tag{4.9}$$

Since

$$- \bar{B}(m^2, \ 0, \ \Lambda^2) > -\bar{B}(m^2, \ k^2, \ \Lambda^2) > 0 \tag{4.10}$$

for k^2 Euclidean, tachyons will be absent if

$$0 < -\bar{B}(m^2, \ 0, \ \Lambda^2) < \left| \frac{1}{\lambda_0} \right| , \tag{4.11}$$

i.e., if

$$\frac{1}{96\pi^2} \left[\ln \left(\frac{\Lambda^2 + \chi}{\chi} \right) - \left(\frac{\Lambda^2}{\Lambda^2 + \chi} \right) \right] < \left| \frac{1}{\lambda_0} \right| . \tag{4.12}$$

Again, a straightforward analysis analogous to that of Abbott, *et al.*, shows that the <u>upper-branch</u> of Fig. 9 <u>always</u> leads to tachyons.

What about the lower-branch of Figs. 8 and 9? In that case the same analysis shows that the tachyons are <u>always</u> absent. In fact the branch-point appears at $\phi^2 = 0$ in Figs. 8 or 9 just when there is a zero-mass O(N) singlet bound state. We return to the case $\lambda_0 < 0$ in Sec. 6 when we discuss a possible double-scaling limit.

V. Physical Properties

A.   The Spectrum

We have seen that when $\lambda_0 > 0$ and $< \Phi_a >^2 = v^2 = -6N\mu^2/\lambda$, the theory has a spontaneously broken symmetry, without tachyons being present. This phase has (N–1) massless Goldstone bosons transforming as an O(N–1) vector. One can explore the spectrum in the O(N–1) singlet sector by continuing det D^{-1} to Minkowski momenta. An O(N–1) singlet resonance, will occur if

$$\left[\frac{1}{\lambda_0} - Re \ \bar{B}(0, -s_r, \Lambda^2) \right] = \frac{1}{3N} \left(\frac{v^2}{s_r} \right) \tag{5.1}$$

has a solution. [If the width of the resonance can be neglected, $\sqrt{s_r}$ will be the mass of the resonance.] Since

$$-\text{Re} \ \bar{B}(0, -s, \Lambda^2) \xrightarrow[s \ll \Lambda^2]{} \frac{1}{96\pi^2} \left\{ \ln \left(\frac{\Lambda^2}{s} \right) + \ldots \right\} > 0 \tag{5.2a}$$

and

$$-\text{Re } \bar{B}(0, -s, \Lambda^2) \xrightarrow[s \to 4\Lambda^2_-]{} \frac{-\Lambda}{96\pi\sqrt{4\Lambda^2 - s}} + \dots \quad < 0 \tag{5.2b}$$

equation (5.1) will always have a solution.

First consider the possibility that $s_r \ll \Lambda^2$. Then the width is negligible, and we estimate

$$s_r \simeq \frac{\lambda_0(v^2/3N)}{\left[1 + \frac{\lambda_0}{96\pi^2}\ln(\Lambda^2/s_r)\right]} \ll \Lambda^2 \tag{5.3}$$

which is consistent if $\frac{\lambda_0}{3N}\left(\frac{v^2}{\Lambda^2}\right) \ll 1$. If further $\frac{\lambda_0}{96\pi^2}\ln\left(\frac{\Lambda^2}{s_r}\right) \ll 1$, then

$$s_r \simeq \frac{\lambda_0\, v^2}{3N} \tag{5.4}$$

which agrees with the weak-coupling, semi-classical evaluation of the scalar mass from (2.1), <u>except</u> that v_0^2 has now been dressed to v^2.

We do not have an analytic estimate of s_r in the general case, so that detailed results depend on numerical evaluation. Notice from (4.8) that $\bar{B}(0, -s, \Lambda^2)$ only depends on (s/Λ^2). Therefore, the bound-state equation depends on three dimensionless parameters (s_r/Λ^2), $(v^2/N\Lambda^2)$, and λ_0, with (5.1) providing one relation between them. According to (2.1) the large N limit is taken by keeping λ_0 and v^2/N fixed. Physically v^2/N characterizes the broken symmetry scale, while Λ^2 characterizes the scale of "new physics." With these fixed, one can explore (s_r/Λ^2) vs. λ_0, as λ_0 varies from weak to strong coupling.

We plot $-96\pi^2 \text{ Re } \bar{B}(0, -s, \Lambda^2)$ versus $x = s/\Lambda^2$ in Fig. 10 as a solid line. In the same figure we also plot $96\pi^2[\frac{1}{3N}(\frac{v^2}{\Lambda^2})\frac{1}{x} - \frac{1}{\lambda_0}]$ for a "typical" case, shown as a dotted line. Changes in λ_0 merely raise or lower the dotted curve. Figure 10 enables us to understand the qualitative behavior of solutions to (5.1). It is clear from the figure that there are always <u>two</u> solutions of (5.1); one with $s_r/\Lambda^2 < 1$, and the other with $\bar{s}/\Lambda^2 > 1$. The solution $\bar{s}$ should <u>not</u> be considered a prediction of the model, as $\bar{s}$ <u>always</u> appears above the cutoff energy of the model. [We comment on the interpretation of this unphysical resonance at the end of this section.] With this understanding, we conclude that the model predicts one resonance with $s_r < \Lambda^2$ in the O(N–1) singlet channel, which is the Higgs meson in typical applications. In the limit where $\frac{\lambda_0}{3N}(\frac{v^2}{\Lambda^2}) \ll 1$, the Higgs mass is given by (5.3) [or (5.4) if appropriate]. Figure 10 makes it clear that as λ_0 is increased, with v^2/N and Λ fixed, the Higgs mass and width increase, but with $s_r < \Lambda^2$ always. However, as λ_0 increases, the width increases as $(s_r)^{3/2}$, so is eventually no longer negligible. We consider this situation in the next subsection.

B.   Scattering Amplitudes

The $\Phi - \boldsymbol{\chi}$ matrix inverse propagator is easily inverted. In the Minkowski region

$$D_{\chi\chi}(0, s^2, \Lambda^2) = \frac{(s/3N)}{s\left[-\frac{1}{\lambda_0} + \bar{B}(0, -s, \Lambda^2)\right] + v^2/3N} \tag{5.5}$$

$$\xrightarrow{s\ll\Lambda^2} \frac{(s/3N)}{s\left\{-\frac{1}{\lambda_0} - \frac{1}{96\pi^2}\left[\ln\left(\frac{\Lambda^2}{s}\right) + i\pi\right]\right\} + v^2/3N} \tag{5.6}$$

A typical example is the meson-meson scattering amplitude in the O(N–1) singlet sector is

$$A(s,t,u) = D_{\chi\chi}(s) + D_{\chi\chi}(t) + D_{\chi\chi}(u) \tag{5.7}$$

which has an s-channel resonance, with a width proportional to $(s_r)^{3/2}$, and a scalar particle exchange in the t and u-channels. In physical applications this would correspond to longitudinal $Z_0 - Z_0$ scattering [8], for example. The amplitudes for the scattering of longitudinal W's for other channels are given by eq. (1) of Naculich and Yuan [7]. If s/Λ^2 is not $\ll 1$, then one must use (5.5) together with (4.8a) continued to Minkowski momenta, rather than the approximate expression (5.6). Then the resonance mass is computed from the position of the peak in Im $D_{\chi\chi}(s)$. A table of resonance masses vs. coupling constant, for $\Lambda = 1$ Tev and $\Lambda = 4$ Tev is given in Table I. The weak-coupling approximation, with the width ignored in (5.1) is satisfactory for $(\lambda_0/16\pi^2) \leq 1.3$ for $\Lambda = 1$ Tev and for $(\lambda_0/16\pi^2) \leq 2.5$ for $\Lambda = 4$ Tev. The width $\Gamma \sim 300$ Gev for $\Lambda = 1$ Tev and $\Gamma = 240$ Gev for $\Lambda = 4$ Tev when $(\lambda_0/16\pi^2) = 0.95$, and varies as $(s_r)^{3/2}$.

From (4.8a). (5.5) and Fig. 10 we see that cross-sections go through a second unphysical resonance. [See Sec. 5a above.] We identify this with the triviality scale $\sqrt{\bar{s}}$ discussed in the literature [5, 6, 7]. In the weak coupling limit, one finds from (5.2b) that $(4\Lambda^2 - \bar{s})/\Lambda^2 \simeq (\frac{\lambda_0}{96\pi})^2$. As λ_0 increases, $\bar{s}$ decreases towards Λ^2 from above, while the Higgs (mass)^2 s_r increases towards Λ^2 from below.

VI. Is There a Double-Scaling Limit?

There has been considerable interest in the double scaling limit for matrix models [9], for which one considers the correlated limit $N \to \infty$ and $g \to g_c$, where g_c is a critical value of a coupling constant. Unfortunately this approach to quantum gravity meets a c=1 barrier. It has been suggested that one consider a double-scaling limit for O(N) models for dimensions D≥2 [10], as the Feynman diagrams of such theories do not fill a surface, but rather have a branched structure. It was hoped that such considerations would give some indications on how the c=1 barrier might be surmounted. However, it has been shown [11] that at the critical point of the D=4 renormalized vector model, the effective potential is everywhere complex, which implies that there is no double-scaling limit for this model. Similar results have been found for D=2 and D=3 [12, 13].

It is interesting to reexamine the possibility of a double scaling limit for the cutoff vector model as an application of our study of the effective potential. We require a zero-mass bound-state in the O(N) singlet channel. An analysis similar to that of Sec. 4A shows that this is not possible for $\lambda_0 > 0$, and therefore we turn to $\lambda_0 < 0$ as in Sec. 4C. A zero-mass O(N) singlet bound-state implies that

$$\frac{1}{\lambda_0} = \bar{B}(\chi,\, 0,\, \Lambda^2) \tag{6.1}$$

$$= -\frac{1}{96\pi^2}\left\{ \ln\left(\frac{\Lambda^2 + \chi}{\chi}\right) - \left(\frac{\Lambda^2}{\Lambda^2 + \chi}\right)\right\} < 0$$

Equation (6.1) can be combined with the gap equation (2.4) to eliminate the logarithm term, with the result

$$\left(\frac{\chi}{\Lambda^2 + \chi}\right) = \left(\frac{96\pi^2}{\Lambda^2}\right)\left(\frac{\mu^2}{\lambda}\right) > 0\;. \tag{6.2}$$

Note that now χ is single-valued just when the theory has a zero-mass O(N) singlet bound-state. The solution to the gap equation is now unique. This occurs precisely where the upper and lower branch of Fig. 9 meet. As a consequence, the effective potential $V(\phi)$ is everywhere complex for this choice of parameters. Therefore, there is no double-scaling limit for the cutoff model for precisely the same reason [11] that a double-scaling limit is absent in the renormalized theory. Our detailed analysis of the effective potential agrees with a general argument of Moshe [13]. The discussion of this section reiterates that the upper-branch of Figs. 8 or 9 always leads to amplitudes with tachyons, while the lower-branch does not, as the zero-mass bound-state divides the two branches.

VII. Concluding Remarks

In this paper we have studied a cutoff version of the O(N) $\lambda\phi^4$ vector model to leading order in the 1/N expansion. An important aspect of our work is a detailed analysis of the effective potential of the model; a topic not studied in previous work. We found that the effective potential was surprisingly rich in structure. Most notably there is a disjoint branch of the effective potential which is everywhere complex, coming from $\chi < -\Lambda^2$ for all values of the parameters of the model, and which has no lowest energy [see Figs. 1 to 5].

It is known for a very long time [14] that a cutoff-model should have some fundamental difficulties, as higher-derivative or non-local cutoff theories suffer either from ghosts and absence of unitarity (e.g., Pauli–Villars theories), or no lowest energy bound. In fact one can convert an indefinite metric theory with ghosts to one with positive metric and no lowest energy by a change in the definition of the adjoint operator, e.g., $A^+ = (-1)^n\, A^*$, where n is the ghost number operator, and $A^+(A^*)$ is the new (usual) adjoint operator. Attempts at making indefinite theories completely consistent [15] have failed [16]. Therefore, any cutoff O(N) vector model should have some fundamental difficulty. The models studied by Heller, et al., [17] have ghosts, but do have a lowest energy bound, while our model does not have ghosts, but fails to have a lowest energy state. The analysis of Pais and Uhlenbeck [14], and the above remarks strongly suggest that these two difficulties are in fact two different aspects of the same problem, which may be related by a change in the adjoint operator, as mentioned above.

An essential result of our cutoff model is that the branch of the effective potential which has no lowest energy and is everywhere complex is disjoint from the rest of the potential, at

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least to leading order in 1/N. It is this feature which allowed us to restrict our attention to those branches of the effective potential for which $\chi > -\Lambda^2$.

Thus analysis of the effective potential showed that one must restrict the composite classical field χ to satisfy $\chi > -\Lambda^2$, where Λ is the fixed cutoff mass-scale of the model, and the bare coupling constant $\lambda_0 > 0$, in order for the model to be consistent to leading order in 1/N. These same constraints guarantee that the model has a phase with spontaneously broken symmetry, free of tachyons. In this phase the O(N–1) singlet channel of meson-meson scattering has two resonances, one physical with $(\text{mass})^2 = s_r < \Lambda^2$, and the other unphysical since its $(\text{mass})^2 = \bar{s} > \Lambda^2$. In applications the physical state is usually identified with the Higgs boson, while we identify the mass of the unphysical state $\sqrt{\bar{s}}$ with the triviality scale, since for $s \geq \bar{s}$ the model is certainly physically unacceptable. That is, features of the model which appear for $s > \Lambda^2$ are not to be considered predictions of the model, but merely artifacts of the cutoff model.

For weak coupling defined by $\frac{\lambda_0}{3N}(\frac{v^2}{\Lambda^2}) \ll 1$, the Higgs boson $(\text{mass})^2$ satisfies $s_r \ll \Lambda^2$, c.f. (5.3). If further, logarithmic corrections are small, then s_r agrees with the semi-classical estimate, but with dressed vacuum expectation value, c.f. (5.4). In weak-coupling, the triviality mass satisfies $\bar{s} \simeq 4\Lambda^2$. As the coupling $\lambda_0 > 0$ increases, the Higgs mass and width increases, with $s_r < \Lambda^2$. At the same time, the triviality mass decreases toward Λ, with $\bar{s} > \Lambda^2$. Thus, we have $0 < s_r < \Lambda^2 < \bar{s} < 4\Lambda^2$.

The behavior of the Higgs mass and width, and triviality mass-scale are in qualitative agreement with previous studies [5, 6, 17]. Certain features of our particular version of the model should be emphasized however. 1) Study of the effective potential leads to the constraints $\chi > -\Lambda^2$ and $\lambda_0 > 0$. 2) The constrained model is tachyon free, and consistent for $s < \Lambda^2$. Thus, we were able to use very general properties of the bubble integral, whereby $\bar{B}(\chi, k^2, \Lambda^2) < 0$ and real for $\chi > -\Lambda^2$ [c.f. (4.2)], to demonstrate the absence of tachyons in the restricted model. This conclusion therefore does not depend on the details of the cutoff procedure.

There are definite advantages in having a tachyon free formulation of the cutoff vector model. For example, one can hope to go beyond the leading order in 1/N, so as to calculate $1/N^2$ corrections. If tachyons are present in the 1/N approximation as isolated unphysical states, they will then appear in loop-corrections in the next order in 1/N. As a consequence, the effective potential will be everywhere complex in non-leading orders of 1/N [c.f. ref. [2] and Root, ref. [1]], rendering the model inconsistent. If tachyons are absent in leading order, then there is the possibility that the theory will remain consistent in higher orders of 1/N. Therefore, a tachyon free formulation of the cutoff O(N) model to leading order in 1/N is very attractive from a theoretical point of view.

There are still some outstanding questions which deserve further scrutiny. It is not known whether the tachyon free ground-states of the $\lambda_0 > 0$ models are metastable or not. To resolve this issue, a calculation at least to the next order in 1/N will be necessary. It is promising that the ground-state which is tachyon free and the branch with $\chi < -\Lambda^2$ (which has no lowest energy) are <u>disjoint</u> in the effective potential. The issue therefore is whether

or not the tachyon free ground-state will decay to the lower branch in a finite order of the 1/N expansion. In this context, recall that the tachyon free vacuum found by Abbott, *et al.*, [2] <u>was</u> metastable in the renormalized theory.

It might be thought that this issue might be avoided by choosing a model with a Pauli–Villars cutoff (say) which does have a lowest energy, but has ghosts. Since the ghosts represent a failure of unitarity, it is not clear how this will be manifest in higher orders in 1/N. As we have already remarked, a model with ghosts or a model with no lowest energy seem to be two different aspects of the same problem of cutoff theories [14]. Therefore, we expect that the issue of a metastable ground-state applies to both versions of the model. It cannot be avoided.

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NOTE ADDED:

After this paper was initially submitted for publication we became aware of the work by Heller, Neuberger, and Vranas [17]. These authors study related cutoff O(N) models in the large N limit. We wish to thank Professor Neuberger for bringing this to our attention. Their model is not identical to the $\lambda\phi^4$ theory considered here, as they add dimension 6 and 8 operators to the theory and modify the kinetic term. They consider a class of Pauli–Villars regularizations and lattice regularizations as well. The phenomenological aspects of our model are consistent with the conclusions of Heller, *et al.* [5, 17]. It should be emphasized that these authors do not study the effective potential or the double-scaling limit, which were presented in this paper. However, it is likely that their analysis of the Higgs mass and triviality mass-scale is numerically more accurate for applications to phenomenology.

Some authors have expressed the opinion that further study of the large N model would be of "progressively diminishing interest." We do not share this view. Our work shows that a better understanding of the model is made possible by investigation of the effective potential, which in turn suggests issues for further study. In fact we believe that even reformulation of known results can lead to new insights.

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$\lambda_0/16\pi^2$	$\sqrt{s_r}$ in Gev	
	for $\Lambda = 1$ Tev	for $\Lambda = 4$ Tev
0.02	100	80
0.2	300	280
0.4	410	390
0.6	510	460
0.95	610	520
1.30	700	560
1.9	860	640
2.5	resonance	660
3.2	not well separated	680
3.8	from triviality resonance	680

Table I

Table Caption: Approximate resonance mass $\sqrt{s_r}$ in Gev vs. coupling constant $\lambda_0/16\pi^2$ for cutoffs $\Lambda = 1$ Tev and $\Lambda = 4$ Tev.

Figure Captions

Fig. 1: Effective potential for $\bar{\lambda}_0 > \lambda_0 > 0$, $\frac{6\mu^2 N}{\lambda} < 0$, and no restriction on the background field χ. The parameter $\bar{\lambda}_0$ specifies an upper-limit for λ_0 for which branch II remains below branch I for all ϕ^2 beyond the broken symmetry vacuum of branch I. Branch I (II) comes from $\chi > -\Lambda^2 (\chi < -\Lambda^2)$. This case describes broken symmetry and weak coupling. Dotted lines indicate regions where the potential is complex.

Fig. 2: Same as Fig. 1, except for strong-coupling $\lambda_0 > \bar{\lambda}_0 > 0$

Fig. 3: Effective potential for $\lambda_0 > 0$, $\frac{6\mu^2 N}{\lambda} > 0$, and no restriction on the background field χ.

Fig. 4: Effective potential for $\lambda_0 < 0$, $\frac{6\mu^2 N}{\lambda} < 0$, and no restriction on the background field χ. Branch I (II) comes from $\chi > -\Lambda^2 (\chi < -\Lambda^2)$, with each split into 2 subbranches.

Fig. 5: Same as Fig. 4, except $\frac{6\mu^2 N}{\lambda} > 0$.

Fig. 6: Effective potential for $\lambda_0 > 0$, $\frac{6\mu^2 N}{\lambda} < 0$, with the restriction $\chi > -\Lambda^2$.

Fig. 7: Same as Fig. 6, but with $\frac{6\mu^2 N}{\lambda} > 0$.

Fig. 8: Same as Fig. 6, except $\lambda_0 < 0$.

Fig. 9: Same as Fig. 7, except $\lambda_0 < 0$.

Fig. 10: Graph of $-(96\pi^2)\mathrm{Re}\bar{B}(0, -s, \Lambda^2)$ versus $x = s/\Lambda^2$, shown as the solid line, the quantity $96\pi^2[\frac{1}{3N}(\frac{v^2}{\Lambda^2})\frac{1}{x} - \frac{1}{\lambda_0}]$ is also plotted as a dotted line for a typical set of values of the parameters. Changes in λ_0 shifts the dotted curve up or down.

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